REALISTIC ESTIMATIONS OF THE EFFECTIVE STABILITY REGION OF THE TROJAN ASTEROIDS

CH. SKOKOS
Research Center for Astronomy, Academy of Athens,
14 Anagnostopoulou str., GR-106 73, Athens, Greece

1. Introduction

The usual approach in analytical studies of the stability of the Trojan asteroids is to consider simple models for the system such as the two dimensional (2D) planar, and the three dimensional (3D) spatial restricted three body problem (RTBP) (Giorgilli et al., 1989; Simó, 1989; Celletti and Giorgilli, 1991; Celletti and Ferrara, 1996). As an example of a more complicated model for the problem we refer to the model developed by Gabern and Jorba (2001) where the effect of Saturn on the motion of the asteroid has been taken into account. The teqniques used in these papers are based in normal forms or first integrals calculations. Roughly speaking one shows that the system admits a number of approximate integrals, whose time variation can be controlled to be small for an extremely long time. In this case we have effective stability, i.e. even when an orbit is not stable, the time needed for it to leave the neighborhood of the equilibrium is larger than the expected lifetime of the physical system studied. This is the basis to derive the classical Nekhoroshev's estimates (Nekhoroshev, 1997). The first result that guaranties the effective stability of real asteroids was provided by Giorgilli and Skokos (1997) for the 2D RTBP. In the present paper we refer to some recent results for the 3D RTBP obtained by Skokos and Dokoumetzidis (2001).

2. Estimating of the size of the effective stability region

We consider the spatial RTBP in particular for the Sun (S), Jupiter (J) and asteroid (A) system. We introduce a uniformly rotating frame (O, q_1, q_2, q_3) so that its origin is located at the center of mass of the Sun-Jupiter system, with the Sun always at the point (μ, 0, 0) and Jupiter at the point ($\mu-1$, 0, 0). The physical units are chosen so that the distance between Jupiter and the Sun is 1, $\mu = 9.5387536 \cdot 10^{-4}$ and the angular velocity of Jupiter is 1. The time unit is $(2\pi)^{-1}\, T_J$, where T_J is the period of the circular motion of Jupiter around the Sun. So the age of the universe is about 10^{10} time units.

In order to bring the Hamiltonian to a form suitable for the application of the normal form scheme we perform a sequence of transformations:

– We introduce a uniformly rotating frame with its origin on the Sun (S) using the generating function $W_3 = -(Q_1+\mu)p_1 - Q_2 p_2 - Q_3 p_3 + \mu Q_2$, where Q_1, Q_2, Q_3, P_1, P_2, P_3 are the heliocentric coordinates.

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- We introduce cylindrical coordinates P, Θ, Z, via the canonical transformation introduced by $W_3 = -P(P_1 cos\Theta + P_2 sin\Theta) - ZP_3$.
- We move the origin of the coordinate system to the Lagrangian point L_4 using the generating function $W_2 = p_x(P-1) + (p_y+1)\Theta - \frac{2\pi p_y}{3} + p_z Z$.
- We expand the resulting Hamiltonian in Taylor series around the point L_4 $(x = y = z = p_x = p_y = p_z = 0)$.
- We introduce a canonical transformation which brings the quadratic part of the Hamiltonian to the diagonal form $H_2 = \sum_{j=1}^{3} \omega_j (x_j^2 + y_j^2)/2$, where x_1, x_2, x_3 are the canonical coordinates, y_1, y_2, y_3 the conjugate momenta and $\omega_1 \simeq 9.967575 \cdot 10^{-1}$, $\omega_2 \simeq -8.046388 \cdot 10^{-2}$, $\omega_3 = 1$.

Then, following Giorgilli et al. (1989) we construct the normal form $Z^{(r)}$ up to order r. So we have $Z^{(r)} = Z_2 + Z_3 + \cdots + Z_r + Y^{(r)}$, where Z_s is a homogeneous polynomial of degree s in the new 'normal variables' x'_1, x'_2, x'_3, y'_1, y'_2, y'_3 and $Y^{(r)}$ is a remainder, actually a power series starting with terms of degree $r+1$. The normal form $Z^{(r)}$ admits three approximate first integrals of the form $I'_j(x', y') = ({x'_j}^2 + {y'_j}^2)/2$ for $j = 1,2,3$, the time variation of which is given by $\dot{I}'_j = [I'_j, Z^{(r)}] = [I'_j, Y^{(r)}]$, which is a power series starting with terms of degree $r+1$. We remark that [,] denotes the Poisson bracket.

The stability of the system is studied in domains of the form:

$$\Delta_{\rho R} = \left\{ (x', y') \in \mathbf{R}^6 \, : \, x'^2_j + y'^2_j \leq \rho^2 R_j^2 \right\} \ , \quad j = 1,2,3 \ , \tag{1}$$

where R_1, R_2, R_3 are arbitrary fixed positive constants, ρ is a positive parameter and x', y' stand for x'_1, x'_2, x'_3 and y'_1, y'_2, y'_3 respectively. The norm $\|f\|_{\rho R}$ of a homogeneous polynomial $f(x', y')$ of degree s in the domain $\Delta_{\rho R}$ does not exceed the quantity:

$$\|f\|_{\rho R} \leq \frac{\rho^s}{2^{s/2}} \sum_{j_1 j_2 j_3 k_1 k_2 k_3} \left| C_{j_1 j_2 j_3 k_1 k_2 k_3} \right| R_1^{j_1+k_1} R_2^{j_2+k_2} R_3^{j_3+k_3} \ , \tag{2}$$

where $C_{j_1 j_2 j_3 k_1 k_2 k_3}$ are the complex coefficients of $f(x', y')$ when f is transformed in complex variables ξ, η via the transformation $x'_j = (\xi_j + i\eta_j)/\sqrt{2}$, $y'_j = i(\xi_j - i\eta_j)/\sqrt{2}$ for $j = 1,2,3$ (Giorgilli and Skokos, 1997).

Suppose that the initial point of an orbit lies in the domain $\Delta_{\rho_0 R}$ for some positive value ρ_0. We fix a larger domain $\Delta_{\rho R}$, with $\rho > \rho_0$, and ask how long the orbit will be confined in the latter domain. We shall refer to this time interval as the escape time τ. Since $\dot{I}'_j = dI'_j/dt$, we get

$$dt \geq \frac{dI'_j}{\sup_{\Delta_{\rho R}} |\dot{I}'_j|} \ , \quad j = 1,2,3 \ , \tag{3}$$

where $\sup_{\Delta_{\rho R}} |\dot{I}'_j|$ is the supremum norm of $\dot{I}'_j$, over the domain $\Delta_{\rho R}$. The problem is how to estimate $\sup_{\Delta_{\rho R}} |\dot{I}'_j|$. To this end, we proceed as follows. Assuming that ρ

is smaller than half of the convergence radius of the remainder $Y^{(r)}$ we can use the approximate estimation

$$\sup_{\Delta_{\rho R}} |\dot{I}'_j| < 2\|[I'_j, Y^{(r)}_{r+1}\|_{\rho R} = 2\rho^{r+1}\|[I'_j, Y^{(r)}_{r+1}\|_R \ , \tag{4}$$

where $Y^{(r)}_{r+1}$ is the first term of the remainder. We can estimate the minimum escape time by integrating both parts of Eq. (3) using also Eq. (4). In order to eliminate the dependence of the escape time on the final domain we fix ρ to be equal to $\lambda\rho_0$, with $\lambda > 1$, so the minimum escape time becomes

$$\tau_{r,\lambda}(\rho_0) = \min_{j=1,2,3} \frac{R_j^2}{2(r-1)\rho_0^{r-1}\|[I'_j, Y^{(r)}_{r+1}]\|_R} \left[1 - \frac{1}{\lambda^{r-1}}\right] \ . \tag{5}$$

The above results have been obtained for the spatial RTBP, but can be easily applied to the planar RTBP by assuming that the asteroid remains on the plane of Jupiter's orbit. As already explained the normal form is obtained as an infinite series so in practice we stop the expansions of the several functions at order $\tilde{r} = 30$ for the 3D case and at $\tilde{r} = 50$ for the 2D case. Since in both cases we compute the first order of the remainder the normal form is constructed up to order 29 for the 3D case and up to order 49 for the 2D case. We also use $\lambda = 1.2$, which means that the radius of the final domain is 20% greater than the radius of the initial domain. In order to optimize the minimum escape time with respect to r we compute $\tau_{r,1.2}(\rho_0)$ via Eq. (5) for r running from 3 to the maximum order $\tilde{r} - 1$, for every value of ρ_0. We choose the optimal order r_{opt} of the expansion as the one that gives the maximum value of the escape time. Thus we get the maximum escape time T as function of only the radius ρ_0 of the initial domain:

$$T(\rho_0) = \max_{3 \le r < \tilde{r}} \tau_{r,1.2}(\rho_0) \ . \tag{6}$$

Assuming as a meaningful time interval for the system the estimated age of the universe, which is in our time units 10^{10}, we can find the value of the radius ρ_0 of the corresponding stability region.

3. Application to real asteroids

In order to apply the above results to the real solar system we examine if 98 real asteroids, which are located near the Lagrangian point L_4 (the ones tested by Giorgilli and Skokos, 1997), are inside the estimated effective stability region. So, using the orbital elements of the asteroids we compute their position in the various coordinate systems introduced in the previous section, define the radii R_1, R_2, R_3 for every asteroid and estimate the corresponding radius ρ_0 of the effective stability region. Then an asteroid is inside the stability region if $\rho_0 \geq 1$. We note that in the 2D case only the values of R_1 and R_2 are used.

In the 2D case we guarantee the effective stability of four real asteroids since they are inside the planar stability region, while in the worst case a factor 27 is needed for the most remote asteroid to be inside this region. The optimal order for all asteroids is $r_{opt} \leq 38$, although the expansion of the normal form was performed up to order 49. So the computation of the normal form to orders higher than 38 does not improve the estimations in the 2D case.

In the 3D case one real asteroid is inside the stability region, while in the case of the most remote asteroid the estimated value of ρ_0 is smaller than 1 by a factor 34. In all cases the optimal order of the normal form is the maximum possible, $r_{opt} = 29$, which means that the results may be improved for higher orders. We remark that one would expect to find fewer asteroids inside the stability region in the 3D case than in the 2D case, since the spatial stability region is projected on a plane in the 2D case and points that are outside the spatial stability region may be projected inside the planar stability region. The above results improve significantly older estimations (Giorgilli et al., 1989; Celletti and Giorgilli, 1991) where no real asteroids were inside the stability region and a factor 3,000 was needed for the most remote asteroid to be inside this region.

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